Causality and negative group delays in a simple bandpass amplifier

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We demonstrate a bandpass amplifier which can be constructed from common electronic components and has the surprising property that the group delay is negative in most spectral regions. A pulse propagating through a chain of such amplifiers is advanced by several milliseconds: the output waveform *precedes* the input waveform. Although striking, this behavior is not in conflict with causality, as demonstrated by experiments with pulses which start or end abruptly.

I. INTRODUCTION

Long ago, Sommerfeld and Brillouin [1] treated the problem of propagation of light in a dispersive medium. They identified five velocities which characterize a narrow-band pulse, among which were the *phase velocity*, at which a null of the carrier wave (or other point of constant phase) travels, the *group velocity*, at which the pulse envelope travels, and the *front velocity*, the propagation speed of an abrupt disturbance, such as the sudden switching-on of a signal. The two other velocities defined by Sommerfeld and Brillouin, the *velocity of energy transport* and the so-called *signal velocity* are more complicated and will not be discussed in this paper. They pointed out that in the region of anomalous dispersion within an absorption line the group velocity can be superluminal, i.e., greater than c, the vacuum speed of light. Garrett and McCumber [2] showed with detailed calculations that a Gaussian pulse can propagate with superluminal group velocity and still suffer little distortion from its initial Gaussian shape. Chu and Wong [3] verified this prediction using picosecond laser pulses propagating near the center of the bound A-exciton line of a GaP:N sample. Garrett and McCumber's work also extended the discussion to include amplifiers, which show superluminal group velocities outside the gain line. We have previously [4] [5] shown that for an amplifier consisting of an inverted two-level medium, the spectral window of superluminal group velocities extends to DC, where pulse reshaping due to group velocity distortion and gain are very small. All of these authors emphasize that superluminal group velocities do not contradict relativistic, or "Einstein," causality; indeed it has been demonstrated [6] that causality *requires* that dispersive media show superluminal group velocities in some part of the spectrum.

Much of the recent work on superluminal group velocities has been associated with studies of quantum tunneling and its electromagnetic analogues [7]. Superluminal group velocities have been observed using microwave pulses [8] [9], and with light passing through a dielectric mirror [10] [11].

These experimental and theoretical studies make clear that superluminal group velocities are in no way unphysical. As argued by several authors following Sommerfeld and Brillouin, it is not the group velocity, but rather the front velocity that must be no greater than c by Einstein causality [12] [13] [14] [15] [16] [17]. Unfortunately, the introduction of an abrupt feature which would serve as an experimentally observable front is difficult and has not been accomplished in the microwave and optical systems. This has led some to question the relevance of the front velocity to recent experiments [18].

In this paper we demonstrate a simple bandpass amplifier which shows similar, seemingly noncausal pulse advances and clearly illustrates the issues of pulse transmission and causality. As an illustration, this electronic system has several advantages over the microwave and optical systems. First, the amplifier is conceptually simple and can be constructed from inexpensive electronic components. In addition to the practical advantages, this allows a clearer exposition of the origin of the pulse advance, without the complications of tunneling or a material medium. Second, the time scale of the amplifier can be made very long. This allows the pulse advances to be perceived in real time, but more importantly it allows experiments with abrupt changes of signal, or "fronts." Finally, at the low frequencies characteristic of the amplifier, delays due to spatial propagation are negligible. For this reason, the problem of relativistic causality, which involves at least one spatial dimension in addition to the time dimension, is reduced to a conceptually simpler problem of causality *per se* which involves only time. Many of the arguments presented here are given for spatially extended systems by Diener [17].

II. BANDPASS AMPLIFIER

A simple bandpass amplifier, shown in Fig. 1, consists of an operational amplifier in a non-inverting feedback configuration with a resonant element (an LRC circuit) in the feedback loop. All components are assumed ideal and their values are time-independent. The ideal components are linear, of course, but it should be stated at the outset that throughout this paper the discussion is limited to linear systems and furthermore that all lengths and associated propagation times are negligibly small. Since the analysis is simplest in the frequency domain,

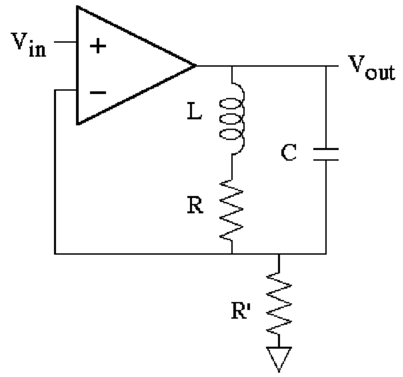

Fig. 1. Bandpass amplifier.

we define the input and output voltages in terms of their Fourier components

$$V_{\rm in}(t) \equiv \int_{-\infty}^{\infty} d\omega \tilde{V}_{\rm in}(\omega) e^{-i\omega t} \tag{1}$$

$$V_{\rm out}(t) \equiv \int_{-\infty}^{\infty} d\omega \tilde{V}_{\rm out}(\omega) e^{-i\omega t}. \tag{2}$$

The output voltage is related to the input by

$$\tilde{V}_{\rm out}(\omega) = T(\omega)\tilde{V}_{\rm in}(\omega) \equiv A(\omega) e^{i\phi(\omega)} \tilde{V}_{\rm in}(\omega) \tag{3}$$

where $A(\omega)$ and $\phi(\omega)$ are real. From the condition that the operational amplifier maintains equal voltages at its inputs, we find the transfer function

$$T(\omega) = 1 + \frac{1}{R'} \frac{1}{(R - i\omega L)^{-1} - i\omega C}. \tag{4}$$

The amplitude and phase of $T(\omega)$ are shown in Fig. 2 for component values $R = 3\,\mathrm{k}\Omega$, $L = 100\,\mathrm{H}$, $C = 0.098\,\mu\mathrm{F}$ and $R' = 30\,\mathrm{k}\Omega$. The curves represent the calculated transfer function while the symbols are measured values from an amplifier described below and in the appendix. Transfer function amplitude and phase were measured by generating sine-wave inputs with a PC-based general-purpose data acquisition board and recording the output with the same board. Although adequate for this purpose, the data acquisition board proved too noisy for the experiments with cascaded amplifiers described below.

III. GROUP DELAY AND GROUP ADVANCE

We consider the propagation of a narrow-band pulse of carrier frequency ω_0 through the amplifier. If the input pulse is described by

$$V_{\rm in}(t) = \int_{\Delta\omega} d\omega \tilde{V}_{\rm in}(\omega) \exp[-i\omega t] \tag{5}$$

where $\Delta\omega$ indicates integration over the narrow frequency range around ω_0 in which the signal amplitude

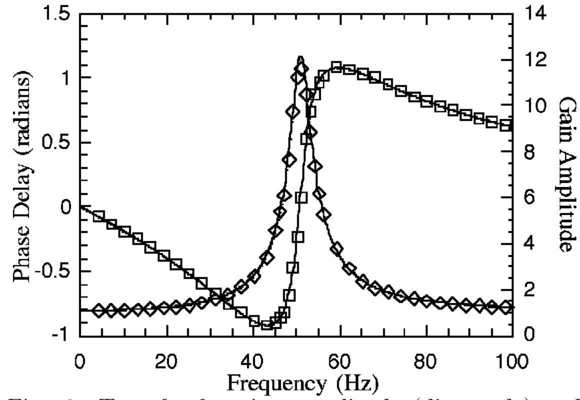

Fig. 2. Transfer function amplitude (diamonds) and phase (squares) for a single amplifier as described in the text. Symbols represent experimental points, curves represent theory. Some points have been omitted for clarity.

is appreciable, then the output is given by the stationary phase approximation

$$\begin{aligned} V_{\rm out}(t) &= \int_{\Delta\omega} d\omega \tilde{V}_{\rm in}(\omega) A(\omega) \exp[-i(\omega t - \phi(\omega))] \\ &\approx A(\omega_0) e^{i\phi(\omega_0)} \\ &\quad \times \int_{\Delta\omega} d\omega \tilde{V}_{\rm in}(\omega) \exp[-i\omega(t - \frac{\partial\phi}{\partial\omega}|_{\omega_0})] \end{aligned} \tag{6}$$

where we have assumed that $A(\omega)$ does not vary appreciably over $\Delta\omega$ and that the expansion of $\phi(\omega)$ can be truncated at first order. If we define the group delay as

$$t_g(\omega_0) \equiv \left.\frac{\partial\phi}{\partial\omega}\right|_{\omega_0} \tag{7}$$

we see that to within our approximations the output is a copy of the input, scaled, phase shifted, and delayed by the group delay

$$V_{\rm out}(t) \approx A(\omega_0) e^{i\phi(\omega_0)} V_{\rm in}(t - t_g(\omega_0)). \tag{8}$$

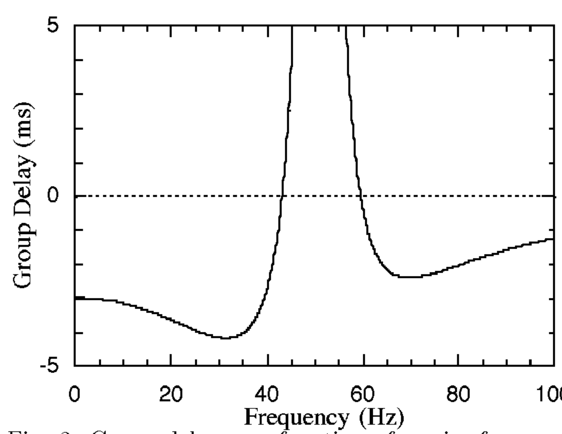

Fig. 3. Group delay as a function of carrier frequency for the amplifier described in the text.

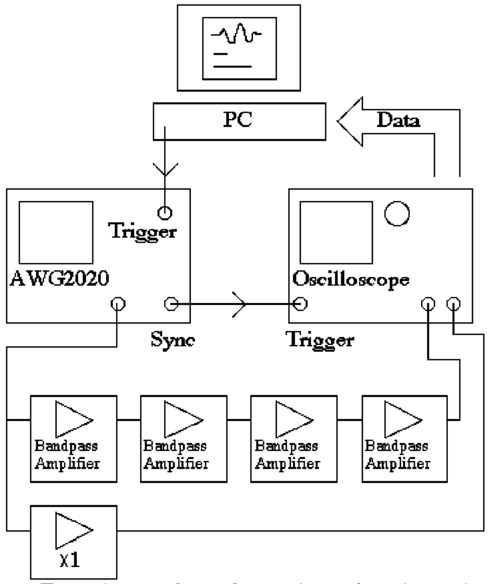

Fig. 4. Experimental configuration. A unity-gain amplifier was included between the AWG2020 and the oscilloscope to prevent feedback.

The group delay as a function of carrier frequency for the bandpass amplifier (same component values as above) is shown in Fig. 3. It is clear from the figure that the group delay is *negative*, i.e., there is a group *advance*, in spectral regions away from the amplification line; the output waveform emerges *before* the input. The origin of this surprising effect is clear if we consider the behavior of the LRC circuit in the feedback loop. The frequency-dependent voltage divider is a passive filter that shows a positive group delay except in the blocked band near the LRC resonance. To maintain equal voltages at its inputs, the operational amplifier must compensate for the filter's group delay by supplying a group advance.

To demonstrate this surprising behavior, we connected four identical amplifiers in series and observed the propagation of signals through the chain. Each amplifier in the chain adds to the group advance, but the peak gain (and hence the output noise) grows geometrically. For this reason it was impractical to add further amplification stages. The signal source was a Tektronix AWG2020 Arbitrary Waveform Generator, and both the input to and the output from the amplifier chain were recorded simultaneously with a real-time oscilloscope. The set-up is shown schematically in Fig. 4. In each run of the experiment, the input pulse was a discrete approximation to a sinusoid with a Gaussian envelope. The time window of the signal source was chosen broad enough that the tails of the Gaussian were clipped only in regions where the signal was smaller than the voltage resolution of the AWG2020.

Typical single oscilloscope traces are shown in Fig. 5 for input pulses with carrier frequencies chosen above resonance, below resonance, and at zero frequency. In each case the peak of the output envelope *precedes* the peak of the input envelope. The ringing which follows the 112 Hz and 25 Hz pulses is not noise, it is the amplification of the small on-resonance component of the input pulse. Since the amplifiers have positive group delay on resonance, this ringing follows the main body of the pulse.

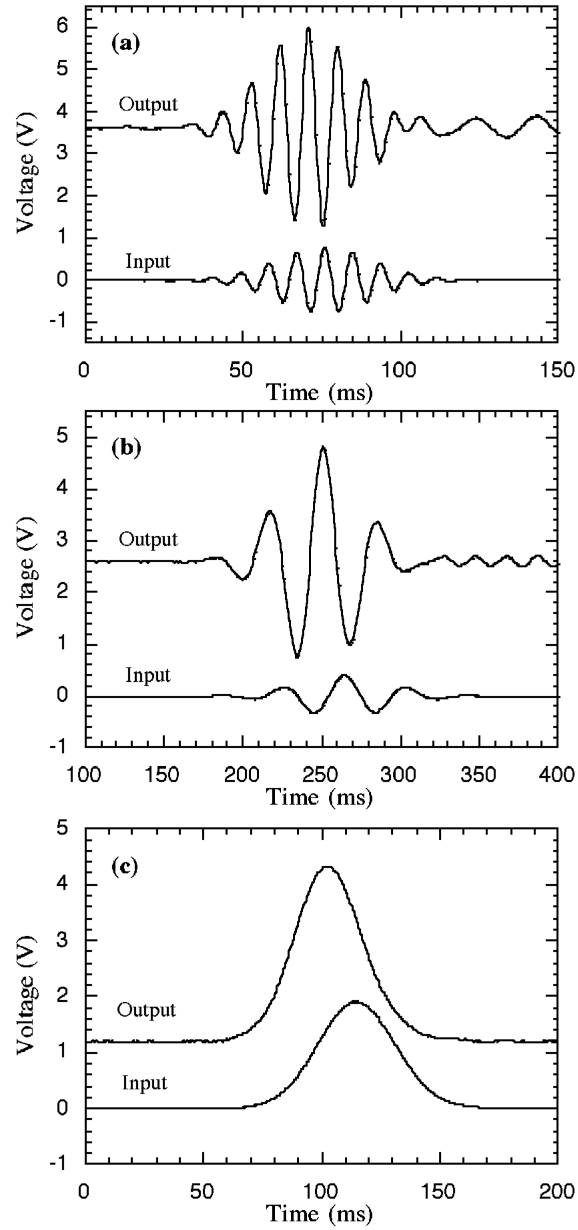

Fig. 5. Input and output to the chain of four amplifiers for pulses with carrier frequency (a) above resonance at 112Hz, (b) below resonance at 25 Hz, (c) zero. Input and output were acquired simultaneously with no averaging. Curves have been offset vertically for clarity.

In contrast to previous experiments which saw pulse advances (relative to propagation in vacuum) on the scale of femtoseconds in the optical experiments and of nanoseconds in the microwave experiments, the pulse advances in Fig. 5 are several milliseconds and can be per-

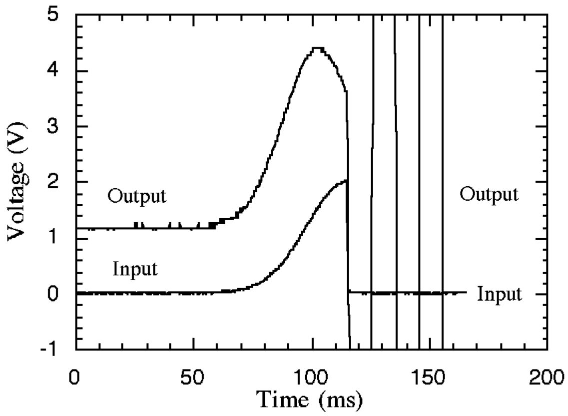

Fig. 6. Input and output to the chain of four amplifiers for a pulse with a "back." Input pulse is the same shape as in Fig. 5 (c) but drops to zero at the midpoint of the pulse. Input and output were acquired simultaneously with no averaging. Curves have been offset vertically for clarity.

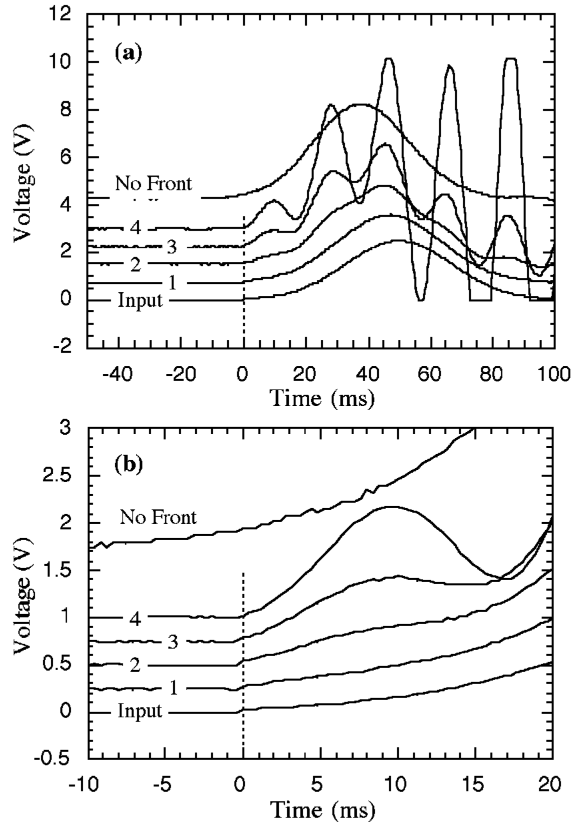

Fig. 7. (a) Propagation of a pulse with a "front." Input pulse is of the same shape as in Fig. 5 (c), except that the leading edge is truncated: the signal is zero before time $t = 0$. Pulse is shown as input and after 1,2,3 and 4 amplifiers. Curves are averages of ten oscilloscope traces. Also shown for comparison is a single trace of the output of the amplifier chain in response to the complete, non-truncated pulse. (b) Closeup near $t = 0$. Curves have been offset for clarity.

ceived in real time by watching traces on an analog oscilloscope. The duration of the pulse advance could in principle be made arbitrarily large if the resonant frequency of the amplifier were reduced while keeping constant the peak gain and the resonator Q.

IV. BACKS AND FRONTS

In the above experiments, the input and output pulses have very similar pulse envelopes. For this reason one might be tempted to conclude that the output pulse is a copy of the input pulse which has been shifted backward in time by the amplifier. This appearance suggests, for example, that the peak of the output is a response to the peak of the input which has not yet arrived. Of course, a causal system cannot respond to something that has not yet happened; the output at some time depends on the input at past and present, not future, times. Specifically, in Fig. 5 (c) the peak of the output is produced in response to earlier input, which does not include the input peak. Whatever merit there may be in associating the input and output peaks, one does not cause the other. This point is made most clearly by sending the early half of a pulse through the amplifier chain. As shown in Fig. 6, such a pulse has the same Gaussian form as the input of Fig. 5 (c) until the midpoint of the Gaussian, and then drops abruptly to zero. Rather than anticipating the arrival of the abrupt drop, the output traces out a smooth peak. The output is the same as it was for a complete Gaussian up until the arrival of the discontinuity (the "back" of the pulse) at which point the amplifiers receive such a kick that they saturate with ringing.

The fact that the output peak is a response to the just the early part of the input suggests an interpretation of the pulse advances of Fig. 5: The amplifier chain amplifies the early part of the input pulse but attenuates the middle and late parts. This reshapes the pulse into something which is still approximately Gaussian in shape and appears shifted backward in time. The group advance continues to accurately describe this shift of the pulse envelope, but this does not imply that any causal effects are going backward in time.

Sommerfeld and Brillouin considered the propagation of a signal with a definite starting point, a "front" before which there was zero signal. By causality, the front must reach the output no earlier than it reaches the input and no signal can precede the front. As the pulse is advanced by an amplifier, it must distort to avoid overtaking the front. It is interesting to illustrate this with a Gaussian pulse that is missing its forward tail, as shown in Fig. 7. Although the main body of the pulse is shifted backward in time by each successive amplifier stage, the pulse becomes progressively more distorted. Of course, no signal precedes the front, which reaches the input and the output of each amplifier at the same time.

V. CAUSALITY OF THE AMPLIFIER

To make rigorous the claim that the amplifier of Fig. 1 is causal, we express the response in the time domain. In the frequency domain the output is the product of the input and the transfer function, so by the *faltung* theorem the output in the time domain is a convolution of the input with the Green's function $G(t)$

$$V_{\rm out}(t) = \int_{-\infty}^{\infty} d\tau G(t-\tau) V_{\rm in}(\tau). \tag{9}$$

The Green's function or "impulse response" describes the output of the amplifier in response to a unit-strength delta function and is given by the Fourier transform of the transfer function

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega T(\omega) e^{-i\omega t}. \tag{10}$$

For the model amplifier with $T(\omega)$ given by equation (4), the Green's function can be calculated exactly. The first term in the integral gives back the original delta function. The second can be evaluated by closing the contour at infinity in the upper half plane for $t > 0$ and in the lower half plane for $t < 0$. This gives the Green's function

$$\begin{aligned} G(t) &= \delta(t) \\ &+ \begin{cases} 0 & t<0 \\ e^{-\gamma t}\frac{1}{R'C}\left[\cos(\omega_r t) + \frac{R}{2L\omega_r}\sin(\omega_r t)\right] & t>0 \end{cases} \\ &\equiv \delta(t) + G'(t). \end{aligned} \tag{11}$$

Here the damping rate is $\gamma = R/2L$ and $\omega_r = \sqrt{1/LC - R^2/4L^2}$ is the ring-down frequency of the amplifier. $G'(t)$ suffers a step discontinuity at $t = 0$ and its value at that point is indeterminate. It can be shown that this point makes no contribution to the convolution integral of equation (9) and hence has no effect on the output voltage. For $t > 0$, the impulse response shows the ring-down behavior of the damped tank circuit in the feedback loop. Most importantly, the Green's function is strictly zero for $t < 0$, a property which allows us to describe the Green's function as causal.

We use the fact that $G(t)$ is zero for $t < 0$ to write the convolution in the manifestly causal form:

$$V_{\rm out}(t) = V_{\rm in}(t) + \int_{-\infty}^{t} d\tau G'(t-\tau) V_{\rm in}(\tau). \tag{12}$$

This makes clear that the output depends only on the present and past values of the input (and not on future values). Of course, since each amplifier in the chain is causal, the behavior of the entire chain is causal as well. In general, the output of any linear system with a causal Green's function can be written in a similar, manifestly causal form.

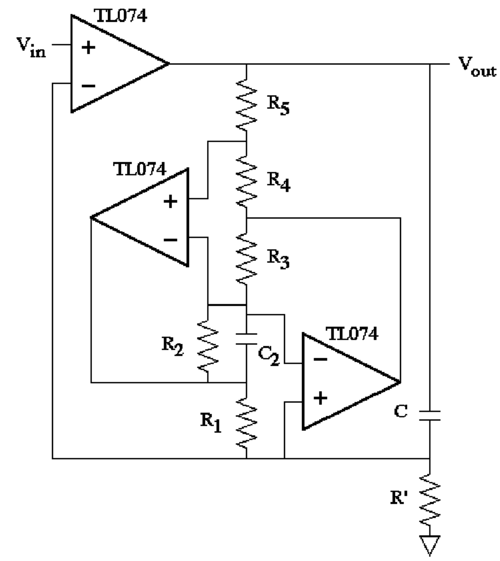

Fig. 8. Gyrator realization of the bandpass amplifier.

Causality demands that no effect, such as the output voltage of the amplifier reaching a particular value, can precede its cause(s). In a single reference frame, precedence is determined simply by the order in which the cause and effect occur. But in special relativity the time ordering of spacelike separated events depends on the reference frame of the observer. If we want causality to hold in all frames, relativistic, or "Einstein," causality must hold: No effect can be caused by an event outside of its past light-cone. In this experiment, the two causality conditions are operationally the same: if one event follows another by the smallest observable time difference, $\Delta t \sim 100\mu$s, then the events would have to be separated by $\Delta x = c\Delta t \sim 30,000$m (much larger than our laboratory) in order to be spacelike separated. If one event noticeably precedes another in the lab frame, it precedes it in all frames.

An alternate method for checking the causality of the amplifier would be to verify that the transfer function satisfies the Kramers-Kronig relations, also known as Bode's relations when applied to electronic circuits [19] [20].

VI. CONCLUSION

The simple bandpass amplifier described above illustrates the paradox of seemingly noncausal pulse advances in a causal system: The amplifier is known to be causal because its Green's function is causal, as shown by exact calculation. In seeming contradiction of this, the amplifier can advance a waveform so that the peak and other features appear at the output *before* the corresponding features arrive at the input. The same amplifier can demonstrate the resolution of the paradox: Experiments on pulses with "backs" and "fronts" make clear that when smooth pulses are advanced by the amplifier, the features of the input waveform do not cause the corresponding features of the output waveform. The amplifier reshapes the input pulse in a causal manner to produce an output which is similar in shape to the input.

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VII. APPENDIX: CONSTRUCTION OF THE AMPLIFIERS

The experienced reader has noticed that the amplifier described above contains one unlikely circuit element, the 100 H inductor. To achieve a low resonance frequency $\omega_r \approx 1/\sqrt{LC}$ it is necessary to have both a large capacitor and a large inductor in the tank circuit. Fortunately, a large inductor can be simulated using a gyrator consisting of two operational amplifiers, capacitors and resistors, as shown in Fig. 8 and described in [21]. Component values are given in Table 1. The impedance of the gyrator is $R_1 R_3 R_5 / Z_2 R_4$ where Z_2 is the impedance of the capacitor and resistor in parallel. This is equivalent to a 100 H inductor and 3 kΩ resistor in series, as in the idealized circuit of Fig. 1. The value of the capacitor used in the text ($C = 0.098\,\mu$F) is a hand-fit to the transfer function data and is well within the manufacturer's stated tolerance of 10%.

Because even a small amount of feedback from the last amplifier in the chain to the first could cause the entire chain to oscillate, a number of measures were taken to ensure isolation of the amplifiers from each other. Each amplifier was placed in a separate shielding box with independent voltage regulators for the supplies to the TL074. All connections from one box to another were made with coaxial cable. A follower (unity gain amplifier) was inserted as a buffer between the AWG2020 and the oscilloscope as shown in Fig. 4 to prevent feedback through the oscilloscope inputs.

R'	30.1 kΩ	R_4	10.0 kΩ
R_1	10.0 kΩ	R_5	33.2 kΩ
R_2	332 kΩ	C	0.10 μF
R_3	30.1 kΩ	C_2	0.10 μF

Table 1. Component values.

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